NON-CLASSICAL LOGICS VS. CLASSICAL

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Abstract. We start with a critique of the basic laws of classical propositional logic**C**. It is worth stressing that there is no law of classical logic, which would not be criticized. We pay attention to the interesting phenomena of extension of classical logic through restriction of its laws, and show that classical logic is only the core (and actually the most simple construction), which naturally gives rise to different and often continuous classes of non-classical logics. We get a continuity logical systems through extension of **C** by the axioms with new primitive symbols, for example the modal operations, or refuting some of the axioms of **C**, for example the law of excluded middle. Also we get continuum of the closed classes of functions adding new truth-values to **C**. The main conclusion is that continuity is distinguishing feature of non-classical logics.

1. Criticism of the Basic Laws of Classical Logic

Criticism of the basic laws and principles of classical logic **C** was already undertaken at the beginning of the 20th century.The most surprising thing in all this is that it took place simultaneously with the construction by A. Whitehead and B. Russell the grandiose building of "Principia Mathematica" (1910 – 1913), with getting up of the first metatheorems of **CL** by D. Hilbert and P. Bernays (see[Zach 1999]) and at the same time with independent publication of metatheorems by Post in 1921 (consistency, deductive completeness, decidability, functional completeness).

First of all, L.E.J. Brouwer in “The Unreliability of the Logical Principles” of 1908 refused the law of the excluded middle $A \vee \neg A$ in mathematical reasoning (see [Brouwer 1975]). Also the law of double negation elimination $\neg\neg A \supset A$ does not hold in general. The idea of Brouwer’s theses was the fact that the laws of classical logic are neither prior nor absolute that, in principle, differed from Frege’s settings. Brouwer’s attitudes in 1930, eventually led to the creation by A. Heyting of intuitionistic logic **H** (see English translation of Heyting’s work in [Mancosu (ed.), 1998]).

In 1910 simultaneously and independently the criticism of the law of contradiction $\neg(A \wedge \neg A)$ was started by J. Łukasiewicz(see [Łukasiewicz 1971]) and N.A. Vasiliev (see [Vasiliev1989]). Pay attention to the Łukasiewicz’s book “On the Principle of Contradiction in Aristotle. A Critical Study”, printed in Polish in 1910 which was republished by J. Woleński in 1987. Then this book was translated into German (1994), French (2000), Italian (2003), and Russian (2012) with my introductory paper and comments. It worth notig that Łukasiewicz’s three-valued logic

$Ł_3$ [1920] refutes both the law of contradiction and the law of excluded middle.

Later, the criticism of the law of contradiction eventually led S. Jaśkowski in 1948 (see [Jaśkowski 1969]) and H. da Costa in 1963 (see [da Costa, 1974]) to the construction of paraconsistent logics that became the basis for inconsistent but non-trivial theories. It follows from this strategy that *the Ex Falso principle* $\neg A \rightarrow (A \rightarrow B)$ does not hold. See also excellent Priest's survey from "Handbook of Philosophical Logic" [Priest2002].

But here we must make one important historical note. In 1925 the greatest Russian mathematician A.N. Kolmogorov in the famous article "Tertium non datur principle" (see [Kolmogorov 1967]). He takes into account Brouwer's criticism of classical logic, but he also rejected another law of classical logic, namely the Ex Falso principle. He argued that 'the axiom now considered does not have and cannot have any intuitive foundation since it asserts something about the consequence of something impossible: we have to accept B if the true judgement A is regarded false'. As a result, Kolmogorov for the first time in the world constructed a paraconsistent logic (the first-order implication-negation fragment).

In 1912 C.I. Lewis [Lewis 1912] builds a new theory of *logical inference* instead of the theory of material (classical) implication, as outlined in the «PrincipiaMathematica». The original motive of Lewis was to get rid of the so-called paradoxes of material implication. Under the latest primarily dealt with formulas

$A \rightarrow (B \rightarrow A)$, *if A is true then it is implied by every B.*

and

$\neg A \rightarrow (A \rightarrow B)$, *if A is false then it implies every B. This is referred to as 'explosion'.*

The paradoxes of material implication arise because of the truth-functional definition of material implication, which is said to be true merely because the antecedent is false or the consequent is true. As a result, material implication was replaced by *strict* implication, which definition later demanded an introduction of the modal operators □ (necessity) and ◊ (possibility). In the treatise [Lewis 1918] the first modal calculus was built, which was denoted as **S3**.

It is worth noting that the law $A \rightarrow (B \rightarrow A)$ attracted special attention. The refuting of this law led to the building of a three-valued logic by B. Sobociński [1952] and the building of infinite-valued logic by Sugihara [1955]. The latter logic is exactly the famous **RM**-system.

In the mid-50's, as a result of criticism of the 'paradoxes' of strict implication:

$$A \rightarrow (B \rightarrow B),$$

any tautology is implied by anything whatsoever, since a tautology is always true, and

$(A \wedge \neg A) \rightarrow B,$

any contradiction implies anything whatsoever, since a contradiction is never true,

there appears a relevant direction in logic led by system **R** of relevant implication and its subsystem **E** of entailment (see in details [Andersen and Belnap 1975]). Note that adding a 'harmless' axiom $A \rightarrow (A \rightarrow A)$ to **R** leads to the logic **RM** with very unusual properties, one of which is the violation of the *variable sharing principle.* This principle says that no formula of the form $A \rightarrow B$ can be proven in a relevance logic if A and B do not have at least one propositional variable in common. It follows from this that formulae

$$A \rightarrow (B \rightarrow B) \text{ and } (A \wedge \neg A) \rightarrow B$$

can't be proven in **R**.

In 1960 Hao Wang paid attention to the fact that the classical first-order logic without the law of contraction

$$(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$$

is solvable (see [Wang1960]). This was implemented in 1971 by V.A. Smirnov (see [Smirnov 1972, ch. 5]). That was the beginning of research in logics without contractions (see [Ono and Komori 1985]).

In 1987, a logic system called 'linear logic' [Girard 1987] appeared, its implicative fragment is **BCI**-logic, i.e. this logic is without the laws of $A \rightarrow (B \rightarrow A)$ and contraction. Linear logic found important applications in computer science ([Troelstra 1992].

We emphasize that the process of criticism of the basic laws of logic **C** lasted for the whole century and was successfully completed by the end of it, and at that time logical laws, that had seemed to be very solid, were rejected. So, the implicational law of transitivity was rejected

$$(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C)) \text{ (see [Tennant 1994]).}$$

Also the implicational law of identity $A \rightarrow A$ did not pass the test of time. Since, according to E. Schroedinger, generally it has no place for micro-objects. Such logics are called 'Schrödinger logics' (see [da Costa and Krause 1994]). There are, of course, many other examples of criticism of logical laws, for example, in quantum logic the distributive law does not hold

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C) \text{ (see [Dalla Chiara 1986]).}$$

Subsequently, critics of basic logical laws became total, and it is worth to say that by the 20th century none of the ever known classical laws remained undoubted.

Eventually this led to the extreme diversity of non-classical directions in logic (Gabbay and Guenthner, eds.1983–1987), (Haack 1996), (Suber 2002), and especially:

Priest G. *An Introduction to Non-Classical Logic: From If to Is*, 2nd ed., CUP, 2008;

Gabbay D. and Guenthner F. (eds.), *Handbook of Philosophical Logic* (Second Edition), Vol. 1 - 17, Dordrecht: Kluwer, 2001-2013.

Unexpected result of this process was the appearing of huge classes of new logical systems. Moreover, in the most cases cardinality of these classes equals to continuum. The first outcome of a similar sort belongs to (Jankov 1968) and concerns a cardinality of the class of extensions of intuitionistic logic. Also there are continual classes of Lewis' modal systems, relevant systems, paraconsistent systems and so on. Now the discovery of the continual classes of logics is the most ordinary thing (Gorbunov and Rybakov 2007).

2. Extension of Classical Logic through Restriction of its Laws

Let us pay attention to the very interesting phenomenon that sometimes refutation of certain laws of classical logic expands it. This primarily refers to the Gödel-Lemmon's axiomatization of Lewis' modal logics, i.e. Lewis' modal system is nothing but an extension of classical logic by means of axioms that define modal operators. Also we have a new axiomatization of relevant logics (see [Meyer 1974]). In relation to this phenomenon, many-valued logics represent particular interest.

Assume that an n-valued logic L_n, $n > 1$, is given by means of truth tables. We shall denote the set of truth values by $V = \{0, 1/(n-1), \ldots, (n-2)/(n-1), 1\}$. Assume $D \subseteq V$, where $1 \in D$ and $0 \notin D$. We shall call D the collection of *designated truth values*. Let the algebra of the logic L_n be $\langle V, \sigma\rangle$, where the signature σ consists of operations on V. The logic L_n is completely determined by the triple $\langle V, \sigma, D\rangle$.

Small Latin letters $x, y, z, \ldots$, shall be used to denote arbitrary truth values. We say that the logic L_n is *truth-complete* iff all J-operators $\{J_i \mid i \in V\}$ are functionally expressible in the signature σ, where (see [Rosser and Turquete 1952])

$$J_i(x) = \begin{cases} 1, \textit{if } x = i \\ 0, \textit{if } x \neq i. \end{cases}$$

A logic L_n is said to be **C**-*extending* iff in L_n one can functionally express the binary operations $\vee$, $\wedge$, $\rightarrow$, and the unary operation $\neg$ (whose restrictions to the subset $\{0, 1\}$ of V coincide with the classical logical operations of disjunction, conjunction, implication, and negation). Note that L_n coincides with classical logic **C** over the set {0, 1}.

An algebra $V = \langle V, \vee, \wedge\rangle$ is called a *quasi-lattice* iff for all $x, y, z \in V$ the following conditions are satisfied (see [Płonka1967]):

1. $x \vee x = x \wedge x = x$ (idempotent laws)
2. $x \vee y = y \vee x$, and $x \wedge y = y \wedge x$ (commutative laws),
3. $x \vee (y \vee z) = (x \vee y) \vee z$, and $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ (associative laws)

We say that a logic is *well-quantified* iff the corresponding algebra $\langle V, \vee, \wedge\rangle$ is a quasi-lattice. Note that the quantifiers ∀ and ∃ can be defined in a logic as generalized conjunction and disjunction iff the logic is well-quantified.

Since every lattice is a quasi-lattice, the class of truth-complete **C**-extending well-quantified logics contains many well-known non-classical logics whose conjunction and disjunction induce a lattice structure, e.g., all the finite-valued logics of Łukasiewicz $\mathbf{Ł}_\mathrm{n}$ which appeared in 1922 (see Łukasiewicz 1970], the functionally complete logics of Post $\mathbf{P}_\mathrm{n}$ (see [Post 1921], the class of functionally precomplete logics of Yablonskii $\mathbf{T}_n$ [Yablonskii 1958], the logics corresponding to the algebras of Moisil [Moisil 1941], many of the significance logics considered by Goddard and Routley [1973], Von Wright's truth logic T'' [1985], and so on.

In our opinion, it is even more interesting to consider examples of truth-complete **C**-extending logics such that the algebra $\langle V, \vee, \wedge\rangle$ is a quasi-lattice but not a lattice: a well known example is Bochvar's three-valued logic $\mathbf{B}_3$ [Bochvar 1938] which is "internal" three-valued logic $\mathbf{B}^\mathrm{I}_3$ (also called *Kleene's weak three-valued logic*) plus assertion operation |– (or J_1). So, it is a logic without the *absorption laws*

$$x \vee (x \wedge y) = x, \text{ and } x \wedge (x \vee y) = x,$$

and without the *contrapositive law* (see axiomatization $\mathbf{B}_3$ in [Finn 1974]). We will return to this logic in the next section.

Now it is the main thing. In [Anshakov and Rychkov1985] the specification of a general, *effective* method for construction of Hilbert-type first-order calculi for any truth-complete **C**-extending well-quantified logic was represented. Note that condition of being well-quantified is not necessary for constructing propositional calculi (especially for this calculi see [Anshakov and Rychkov 1994].

All this means that although most many-valued logics was intended to refute one or another law of classical logic, nevertheless a number of such logics are extensions of **C**.

Now in the example of three-valued logic we can establish the *fundamental* difference of this logic from classical.

3. Continuum property of three-valued logics

Let P_n be the set of all n-valued functions defined on the set V_n.[1] The set of functions P_n corresponds to Post n-valued logic $\mathbf{P_n}$ (see Post [1921]. Then, a pair (P_n, C), where C is the operation of *superposition* of functions, is a functional system. Roughly speaking, the result of superposition of functions $f_1, \ldots, f_k$ is the function obtained from $f_1, \ldots, f_k$ either (1) by substituting some of these functions for arguments of $f_1, \ldots, f_k$ or (2) by renaming arguments of $f_1, \ldots, f_k$ or by both (1) and (2).

Let $F \subseteq P_n$. The operation of superposition leads to a closure operator [ ] on the power-set of P_n; intuitively, [F] is the class of all superpositions of functions from F. A class F of functions is said to be closed if $F = [F]$. A class F of functions is functionally complete in P_n, if $[F] = P_n$. A class F is called *precomplete* in P_n, if $[F] \neq P_n$ and $[F \cup \{f\}] = P_n$, where $f \in P_n$ and $f \notin F$ (in other teminology, a precomplete class of functions is called *maximal clone*).

We are interested in investigation of cardinalities of sets of closed functional classes in different logics. The research in this field was started by E. Post. Thus, in [1941] he proved that classical logic only contains an *enumerable* set of different closed functional classes, and he described a lattice of these classes. In 1959 Yu.I. Yanov and A.A. Muchnik for the first time showed that for every $n \geq 3$ the n-valued Post's logic P_n contains *a continuum of distinct closed classes*, that is even P_3 already has a continuum of closed classes (clones on $\{0, {}^1/_2, 1\}$).[2]

Since P_3 has a continuum of closed classes, it is interesting to ask what cardinality is the set of closed classes of other three-valued logics. Of special interest, because of her close connection to the intuitionistic propositional logic **H**, in this respect is the Heyting's three-valued logic $\mathbf{G_3}$ (1930), also called Jaśkowski's first matrix [Jaśkowski 1936] with $\sigma = <\vee, \wedge, \Rightarrow, \neg>$, where

$$x \vee y = max(x, y), \quad x \wedge y = min(x, y).$$

$$x \Rightarrow y = \begin{cases} 1, if\ x \leq y \\ y, if\ x > y, \end{cases}$$

$$\neg(1) = 0, \neg({}^1/_2) = 0, \neg 0 = 1.$$

M.F. Ratsa in [1982] showed that 3-valued logic of Heyting $\mathbf{G_3}$ contains a continuum of distinct closed classes. Since $\mathbf{G_3} \subset \mathbf{Ł_3}$, then $\mathbf{Ł_3}$ also contains a continuum of distinct closed classes.

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[1] The standard definition of Post's n-valued matrix logics looks like as follows. A matrix of the form $\mathfrak{M}_n = <V_n, \neg, \vee, \{n\text{-}1\}>$ is called a *Post n-valued matrix* ($n \in N$, $n \geq 2$) provided that $V_n = \{0, 1, 2, ..., n\text{-}1\}$; $\neg x = x + 1 (mod\ n)$, $x \vee y = max(x, y)$, and $\{n\text{-}1\}$ is the set of the designated elements of $\mathfrak{M}_n$.

[2] See also [Hulanicki and Swierckowski 1960].

Now let us consider the following consequence of logics:

$$\mathbf{B}_3 \subset \mathbf{E}_3 \subset \mathbf{Ł}_3 \subset \mathbf{P}_3,$$

Where $\mathbf{E}_3$ is Ebbinghaus' three-valued logic [Ebbinghaus 1969], and $\mathbf{B}_3$ is Bocvar's three-valued logic with $\sigma = <\cup, \cap, \sim, |\!-\!>$, where $<\cup, \cap>$ is a quasi-lattice (see above), $<\cup, \cap, \sim>$ is *Kleene's weak three-valued logic* [1938], and $|\!-$ is:

$$|\!-(1) = 1, |\!-({}^1/_2) = 0, |\!-(0) = 0.$$

The key feature of $\mathbf{B}_3$ is the following, which can be easily proved by induction on the complexity of the formula φ:

> *Let φ be a formula of $\mathbf{B}_3$ without $|\!-$ and let v be a valuation of $\mathbf{B}_3$. Then: $v(\varphi) = {}^1/_2$ iff $v(p) = {}^1/_2$ for some propositional variable p.*

We see that $\mathbf{B}_3$ is a functionally weak logic. For a long time the question about the properties of $\mathbf{B}_3$ has been open. But in this year my pupil Nikolay Prelovskiy proved my hypothesis [Karpenko 2010] that the set of closed classes in Bochvar's 3-valued logic $\mathbf{B}_3$ has the power of continuum.

That's not all. Note that Segerberg's three-valued nonsense logic $\mathbf{S}_3$ [1965] is the same as $\mathbf{B}_3$ but with two designated values. K. Segerberg also considered Halldén's three-valued logic $\mathbf{H}_3$ [Halldén 1949] with $\sigma = <\cup, \cap, \sim, \nabla>$ (in our terms), where

$$\nabla(1) = 0, \nabla({}^1/_2) = 1, \nabla(0) = 0, \text{ or } \nabla(x) \text{ is } J_{i/2}(x).$$

It is important that $\mathbf{H}_3$ is not *truth-complete* (in different from $\mathbf{B}_3$), and therefore

$$\mathbf{H}_3 \subset \mathbf{B}_3$$

So, $\mathbf{H}_3$ is a very functionally weak logic. Nevertheless in [Prelovskiy 2013] it was proven that the set of closed classes in $\mathbf{H}_3$ has the power of continuum!

Situation is the same as with the continual classes of non-classical logics: at the beginning, this is a rare and surprising case, then continuity becomes commonplace. However, in the case of cardinalities of sets of closed functional classes in different logics we can supply the problem of criteria of continuity for such logics. Now my pupils intensively work in this field.

In conclusion, I want to say that continuity is the distinguishing feature of non-classical logics.

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