# Kleene logic and inference

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- critical discussion of 1974 Cleve's attempt to define a truth-preserving consequence relation within a model theoretic framework for inexact predicates
- showing a radically different and sound ground for Kleene inference supported by two "idealization" operators which convert neutral propositions into false or true

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- is inspired by the studies of the foundations of mathematics, especially by the problems of algorithms, cf. also Kleene [1952]
- is based on the third category of propositions, whose logical value of truth or falsity is not essential, undefined or undetermined by means of accessible algorithms

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| $\alpha$ | $\neg\alpha$ |
|---|---|
| f | t |
| u | u |
| t | f |

| $\rightarrow$ | f | u | t |
|---|---|---|---|
| f | t | t | t |
| u | u | u | t |
| t | f | u | t |

| $\vee$ | f | u | t |
|---|---|---|---|
| f | f | u | t |
| u | u | u | t |
| t | t | t | t |

| $\wedge$ | f | u | t |
|---|---|---|---|
| f | f | f | f |
| u | f | u | u |
| t | f | u | t |

| $\equiv$ | f | u | t |
|---|---|---|---|
| f | t | u | f |
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| $\rightarrow$ | f | u | t |
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| t | f | u | t |

| $\vee$ | f | u | t |
|---|---|---|---|
| f | f | u | t |
| u | u | u | t |
| t | t | t | t |

| $\wedge$ | f | u | t |
|---|---|---|---|
| f | f | f | f |
| u | f | u | u |
| t | f | u | t |

| $\equiv$ | f | u | t |
|---|---|---|---|
| f | t | u | f |
| u | u | u | u |
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- $K_3 = (\{\text{f, u, t}\}, \neg, \rightarrow, \vee, \wedge, \{\text{t}\})$ *Kleene matrix*

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- We may, therefore, conclude that Kleene treated the added logical value as apparent or as pseudo-value, distinct from the real truth-values, cf. Turquette [1963].

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- the algebra of inexact classes leads naturally to the three-valued logic $K3$
- Basing on it Cleave [1974] worked out a model-theoretic framework for the logic of inexact predicates and he formulated a notion of logical consequence.

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  as follows: $\alpha$ is a $C3$ consequence of the set of formulas $X$, $X \models_{C3} \alpha$ , whenever for every valuation $v$ of the language in $K3$, $min\{v(\beta) : \beta \in X\} \leqslant v(\alpha)$.

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- His interesting trial ends negatively: "The classical two-valued logic as an instrument of deduction, however, presupposes that neutral propositions are treated as if they were true, and inexact predicates as if they were exact. ..."
- This obviously means that the use of the classical logic for drawing empirical conclusions from empirical premises is unsound.

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The relation of empirical inference:

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The letters chosen for these connectives may be read as Weak and Strong. Let us note that the empirical inference just defined may be also interpreted as a relation which holds between a set of premises X and a conclusion whenever it is true independently from accepted idealisation(s) of empirical sentences in X.

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- $(e_3)$ $\alpha \vdash_{K3} S(\alpha)$ and $W(\alpha) \vdash_{K3} \alpha$.

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The first property is an *inferential modus ponens*, the second a *conditional deduction theorem*. The inferences in $(e_3)$ state that from a sentence its strong idealisation follows while the sentence itself is inferred from its weak idealisation.

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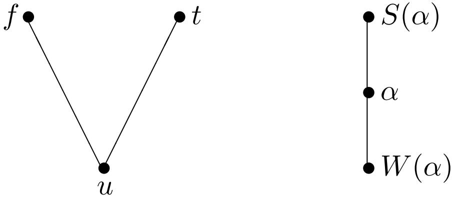

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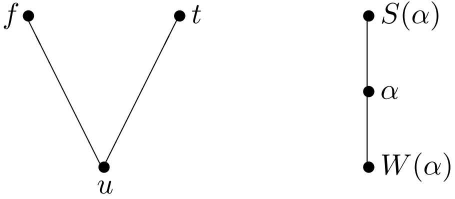


So, any sentence $\alpha$ is deductively situated between its two idealizations: $W(\alpha) \vdash_{K3} \alpha \vdash_{K3} S(\alpha)$.

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- Kleene's case: $A = (\{\text{f, u, t}\},\ \neg,\ \rightarrow,\ \vee,\ \wedge,\ \equiv,\ S,\ W)$, D* $= \{f\}$ and D $= \{t\}$.
- The Kleene q-matrix consequence is a $W$ and $S$ compatible inference from non-rejected premises to accepted conclusions.

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